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Bose Quantization of a New Spin-$\frac{1}{2}$ Equation*

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Abstract

We second quantize a relativistic Schrödinger equation involving a Hamiltonian H that describes free spin-$\frac{1}{2}$ particles and that depends on a parameter G. We require a positive definite metric and a positive definite energy in the Fock space in which the field $\psi(\mathbf{x}, t)$ and its adjoint operate. If $G = \pm i$, one obtains the usual second-quantized Dirac theory, but for real values of G one has Bose statistics. Whereas the anticommutator $[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')]_+$ vanishes for a Dirac field when the interval between $(\mathbf{x}, t)$ and (x', t') lies outside the light cone, when G is real the commutator $[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')]_-$ vanishes for such points.

1. *Introduction*

In an earlier paper (Guertin, 1975b), we studied a Schrödinger equation involving a Hamiltonian that is a second-order differential operator, describes free spin-$\frac{1}{2}$ particles with both energy signs and a definite mass, and depends on a parameter G. By setting $G = \pm i$ one obtains the usual Dirac Hamiltonian, but for real values of G the one-particle theory possesses an indefinite metric; thus, negative energy states have a negative normalization, as in the Sakata-Taketani spin-0 and spin-1 Hamiltonian theories (Sakata & Taketani, 1940; Heitler, 1943) and their arbitrary spin generalizations (Guertin, 1974, 1975a). In this paper we second-quantize the theory using Bose statistics for real values of G and find that, for such values of G, the commutator of the field and its adjoint vanishes when the interval between their space-time arguments is spacelike.

The Hamiltonian of interest is the four by four matrix operator (Guertin, 1975b)

$$H = (\rho_3 + i\rho_2)(1 + G^2)(p^2/2m) + iG\rho_1 \boldsymbol{\sigma} \cdot \mathbf{p} + \rho_3 m \tag{1.1}$$

where G is a constant that is either real or is equal to $\pm i$, $\mathbf{p} = -i\boldsymbol{\nabla}$, $p = |\mathbf{p}|$,

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and $m > 0$ is real. It satisfies the relativistic energy-momentum relation

$$H^2 = E^2 \tag{1.2a}$$

where

$$E = (p^2 + m^2)^{1/2} \tag{1.2b}$$

and describes free spin-$\frac{1}{2}$ particles having both energy signs. Furthermore,

$$H = MH^\dagger M \tag{1.3}$$

where one employs the positive definite metric $M = I$ if $G = \pm i$, in which case H is just the Dirac Hamiltonian, but uses the indefinite metric $M = \rho_3$ if G is real. The wave function $\psi(\mathbf{x}, t)$ is assumed to satisfy the Schrödinger equation

$$i\partial\psi/\partial t = H\psi \tag{1.4}$$

Although (1.4) cannot be manifestly covariant for real values of G, it was demonstrated that it can be made invariant under proper orthochronous Poincaré transformations. Under a space translation $\mathbf{d}$ the wave function goes into

$$\psi'(\mathbf{x}, t) = (1 - i\mathbf{p}\cdot\mathbf{d})\,\psi(\mathbf{x}, t) \tag{1.5a}$$

and under a time translation D it goes into

$$\psi'(\mathbf{x}, t) = (1 + iDH)\,\psi(\mathbf{x}, t) \tag{1.5b}$$

As a result of a rotation by an angle $|\boldsymbol{\theta}|$ about the direction $\boldsymbol{\theta}/|\boldsymbol{\theta}|$ one has

$$\psi'(\mathbf{x}, t) = (1 - i\boldsymbol{\theta}\cdot\mathbf{J})\,\psi(\mathbf{x}, t) \tag{1.6a}$$

where the angular momentum $\mathbf{J}$ has the form

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} + i\tfrac{1}{2}\boldsymbol{\sigma} \tag{1.6b}$$

Finally, under an infinitesimal boost one obtains

$$\psi'(\mathbf{x}, t) = (1 + i\boldsymbol{\lambda}\cdot\mathbf{K})\,\psi(\mathbf{x}, t) \tag{1.7a}$$

where $\boldsymbol{\lambda}$ is a real parameter and where $\mathbf{K}$, the boost generator, can be written

$$\mathbf{K} = \tfrac{1}{2}[\mathbf{x}, H]_+ + \boldsymbol{\Gamma} \tag{1.7b}$$

Here $\boldsymbol{\Gamma} = 0$ for $G = \pm i$, the familiar result for the Dirac theory (Foldy, 1956, Fuschich *et al.* 1971; Kolsrud, 1971; Guertin, 1974, 1975a). But, for real values of G the operator $\boldsymbol{\Gamma}$, a complicated function of the mass, the momentum, the spin, and the ρ matrices, is not even uniquely determined; the freedom in choosing two other real parameters $\theta_\pm(p)$ on which $\boldsymbol{\Gamma}$ depends allows an infinite number of possibilities for $\boldsymbol{\Gamma}$, none of which, in contrast to H, appears to be a local operator.[1] If one could, at least for one value of G, find functions $\theta_\pm(p)$ such that $\boldsymbol{\Gamma}$ is a local operator, this would yield a criterion for uniquely

[1] See equations (6.1b) and (6.5) of Guertin (1975b).

determining $\mathbf{\Gamma}$, but the author has not been able to demonstrate that it is possible to do so.

It has been shown (Guertin, 1975b) that the theory is charge conjugation invariant. Under this symmetry operation $\psi(\mathbf{x}, t)$ goes into

$$\psi'(\mathbf{x}, t) = B\psi^*(\mathbf{x}, t) \tag{1.8}$$

where B may be set equal to

$$B^{(1)} = \rho_2 c \tag{1.9a}$$

if $G = \pm i$ and to

$$B^{(2)} = \rho_1 c \tag{1.9b}$$

if G is real. Here

$$c = \exp(-i\pi\tfrac{1}{2}\sigma_2) \tag{1.10}$$

We also saw that the theory is invariant under the CPT transformation, in which case $\psi(\mathbf{x}, t)$ goes into

$$\psi'(\mathbf{x}, t) = \rho_1\psi(-\mathbf{x}, -t) \tag{1.11}$$

In addition, we discussed the reasons why we do not choose to interpret the theory as being separately invariant under space inversion and under charge conjugation for real values of G, even though both are symmetries for the Dirac theory.

The solution $\psi(\mathbf{x}, t)$ of (1.1) and (1.4) is found in Section 2 of this paper, and the result is second quantized in Section 3. It is found that, in order for the free particle and antiparticle states created from the vacuum to have both a positive definite metric and a positive definite energy, one must employ Bose statistics when G is real, even though Fermi statistics is required for the Dirac theory. Finally, in Seciton 4 it is demonstrated that for real values of G the commutator $[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')]_-$ vanishes for a spacelike interval between $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$, in contrast to the vanishing of the anticommutator $[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')]_+$ for such points when $G = \pm i$. We cannot conclude that this result actually leads to a *local* theory for real values of G, because we are unable to demonstrate at this time that one can actually construct from the field and its adjoint observables that commute when the interval between their space-time arguments lies outside the light cone.

2. *Solutions of the Schrödinger Equation*

To construct explicit solutions of the Schrödinger equation (1.4) in the single particle theory it is useful to know the generalized Foldy–Wouthuysen (FW) operator for the Hamiltonian (1.1). This operator, W, is such that (Guertin, 1975b).

$$H = WH_F W^{-1} \tag{2.1a}$$

where

$$H_F = \rho_3 E \tag{2.1b}$$

is the Hamiltonian in the Foldy canonical representation (Foldy, 1956). Furthermore,

$$W^{-1} = MW^\dagger M \tag{2.2}$$

In writing down the most general form of the generalized FW operator, it is convenient to note that (Guertin, 1975b)

$$H = \sum_\mu H_\mu \Lambda_{\hat{p}}{}^\lambda \tag{2.3a}$$

where

$$H_\pm \equiv (\rho_3 + i\rho_2)(1 + G^2)(p^2/2m) \pm iG\rho_1 p + \rho_3 m \tag{2.3b}$$

and where

$$\Lambda_{\hat{p}}{}^\pm = \tfrac{1}{2}(1 \pm \boldsymbol{\sigma}\cdot\hat{p}) \tag{2.4}$$

with $\hat{p} = \mathbf{p}/p$, are helicity projection operators. Then W can be written[2]

$$W = \sum_\mu W_\mu \Lambda_{\hat{p}}{}^\mu \tag{2.5}$$

where

$$W_\pm = \Omega_\pm \exp(-i\theta_\pm \rho_3) \tag{2.6a}$$

with

$$\Omega_\pm = m\{E(E+m)[(1+G^2)E + (1-G^2)m]\}^{-1/2}(E + H_\pm\rho_3) \tag{2.6b}$$

If $G = \pm i$, in which case one has the Dirac theory, then θ_+ and θ_- are equal and independent of p and may be set equal to zero, but if G is real they may depend on p and are subject only to the restrictions

$$\theta'_+(p) = -\theta'_-(p) \tag{2.7a}$$

$$\theta'_\pm(0) = 0 \tag{2.7b}$$

$$\theta_+(0) = \theta_-(0) \tag{2.7c}$$

We are free to adjust the phase so that $\theta_\pm(0) = 0$ and to define a new function

$$\bar{\theta}(p) = \pm\theta'_\pm(p) \tag{2.8a}$$

with the property

$$\bar{\theta}(0) = 0 \tag{2.8b}$$

Then,

$$W_\pm = \Omega_\pm \exp(\mp i\bar{\theta}\rho_3) \tag{2.9}$$

[2] The functions $\theta_\pm(p)$ that appear in the expression for W are the same ones that we mentioned in Section 1 while discussing the boost generator.

The Foldy canonical wave function ψ^F is related to the wave function ψ in (1.4) as follows:

$$\psi = W\psi_F \tag{2.10}$$

and one can write

$$\psi^F = \psi_+{}^F + \psi_-{}^F \tag{2.11a}$$

where

$$\psi_\pm{}^F = \tfrac{1}{2}(1 \pm \rho_3)\psi^F \tag{2.11b}$$

is such that $\psi_+{}^F$ contains only positive energies and $\psi_-{}^F$ only negative energies. Similarly, as a result of (2.1a) and the above,

$$\psi = \psi_+ + \psi_- \tag{2.12a}$$

where

$$\psi_\pm = \tfrac{1}{2}(1 \pm H/E)\psi \tag{2.12b}$$

is such that

$$\psi_\pm = W\psi_\pm{}^F \tag{2.12c}$$

The explicit construction of solutions to (1.4) involves the same procedure employed for other Hamiltonians (Guertin, 1975a). One may write

$$\psi_\pm(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}}\int d^3q \sum_\sigma u_\pm{}^{(\sigma)}(\mathbf{q})a_\pm{}^{(\sigma)}(\mathbf{q})e^{i(\mathbf{q}\cdot\mathbf{x} \mp E(\mathbf{q})t)} \tag{2.13}$$

where

$$u_\pm{}^{(\sigma)}(\mathbf{q}) = W(\mathbf{q})\tfrac{1}{2}(1 \pm \rho_3)\chi^{(\sigma)} \tag{2.14a}$$

Here, $\chi^{(\sigma)}$, where $\sigma = \pm\frac{1}{2}$, is a four-component column matrix whose elements, labeled by $\mu = 1, \ldots, 4$ satisfy

$$\chi_\mu{}^{(\sigma)} = \delta_{\sigma,3/2-\mu} + \delta_{\sigma,7/2-\mu} \tag{2.14b}$$

The normalization is

$$u_\pm{}^{(\sigma')\dagger}(\mathbf{q})Mu_\pm{}^{(\sigma)}(\mathbf{q}) = \begin{cases} \delta_{\sigma\sigma'} & \text{if } M = I \\ \pm\delta_{\sigma\sigma'} & \text{if } M = \rho_3 \end{cases} \tag{2.15a}$$

$$u_\mp{}^{(\sigma')\dagger}(\mathbf{q})Mu_\pm{}^{(\sigma)}(\mathbf{q}) = 0 \tag{2.15b}$$

Let us recall that the scalar product is

$$\langle \psi, \psi \rangle_M = \int d^3x \psi^\dagger(\mathbf{x}, t)M\psi(\mathbf{x}, t) \tag{2.16a}$$

and that the expectation value of any observable $\mathcal{O}$ is

$$\langle\psi,\mathcal{O}\psi\rangle_M = \int d^3x\psi^\dagger(\mathbf{x},t)M\mathcal{O}\psi(\mathbf{x},t) \tag{2.16b}$$

When one evaluates (2.16a) the result is

$$\langle\psi,\psi\rangle_M = \begin{cases} \int d^3q \sum_\sigma [a_+^{(\sigma)*}(\mathbf{q})a_+^{(\sigma)}(\mathbf{q}) + a_-^{(\sigma)*}(\mathbf{q})a_-^{(\sigma)}(\mathbf{q})] & \text{if } M = I \\ \int d^3q \sum_\sigma [a_+^{(\sigma)*}(\mathbf{q})a_+^{(\sigma)}(\mathbf{q}) - a_-^{(\sigma)*}(\mathbf{q})a_-^{(\sigma)}(\mathbf{q})] & \text{if } M = \rho_3 \end{cases} \tag{2.17a}$$

and one also finds that

$$\langle\psi,H\psi\rangle_M = \begin{cases} \int d^3qE(\mathbf{q}) \sum_\sigma [a_+^{(\sigma)*}(\mathbf{q})a_+^{(\sigma)}(\mathbf{q}) - a_-^{(\sigma)*}(\mathbf{q})a_-^{(\sigma)}(\mathbf{q})] & \text{if } M = I \\ \int d^3qE(\mathbf{q}) \sum_\sigma [a_+^{(\sigma)*}(\mathbf{q})a_+^{(\sigma)}(\mathbf{q}) + a_-^{(\sigma)*}(\mathbf{q})a_-^{(\sigma)}(\mathbf{q})] & \text{if } M = \rho_3 \end{cases} \tag{2.17b}$$

In considering charge conjugation invariance [see equations (1.8)-(1.10)], it is useful to know certain properties of the momentum space solutions $u_\pm^{(\sigma)}(\mathbf{q})$, where the subscript designates the sign of the energy. According to (2.5)-(2.7) and (2.14),

$$Bu_+^{(\sigma)*}(\mathbf{q}) = \eta \sum_{\sigma'} c_{\sigma\sigma'}^{-1} u_-^{(\sigma')}(-\mathbf{q}) \tag{2.18a}$$

$$Bu_-^{(\sigma)*}(\mathbf{q}) = \eta^* \sum_{\sigma'} c_{\sigma\sigma'}^{-1} u_+^{(\sigma')}(-\mathbf{q}) \tag{2.18b}$$

where

$$\eta = \begin{cases} i & \text{if } G = \pm i \\ 1 & \text{if } G \text{ is real} \end{cases} \tag{2.19}$$

This result will be employed in the next section.

3. *Second Quantization*

To second quantize the theory developed so far, it is first convenient to make the identification

$$a_+^{(\sigma)}(\mathbf{q}) = a^{(\sigma)}(\mathbf{q}) \tag{3.1a}$$

$$a_+^{(\sigma)*}(\mathbf{q}) = a^{(\sigma)*}(\mathbf{q}) \tag{3.1b}$$

$$a_-^{(\sigma)}(\mathbf{q}) = \sum_{\sigma} c_{\sigma\sigma'} b^{(\sigma')*}(-\mathbf{q}) \tag{3.1c}$$

$$a_-^{(\sigma)*}(\mathbf{q}) = \sum_{\sigma'} c_{\sigma\sigma'} b^{(\sigma')}(-\mathbf{q}) \tag{3.1d}$$

where an asterisk now denotes Hermitian conjugation in the Fock space in which these operators act. We also write

$$u_+^{(\sigma)}(\mathbf{q}) = u^{(\sigma)}(\mathbf{q}) \tag{3.2a}$$

$$u_{-}^{(\sigma)}(\mathbf{q}) = \sum_{\sigma'} c_{\sigma\sigma'} v^{(\sigma')}(-\mathbf{q}) \tag{3.2b}$$

and from (2.12) and (2.13) we have for the field operator

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3q \sum_{\sigma} [u^{(\sigma)}(\mathbf{q}) a^{(\sigma)}(\mathbf{q}) e^{i[\mathbf{q}\cdot\mathbf{x} - E(\mathbf{q})t]} + v^{(\sigma)}(\mathbf{q}) b^{(\sigma)*}(\mathbf{q}) e^{-i[\mathbf{q}\cdot\mathbf{x} - E(\mathbf{q})t]}] \tag{3.3}$$

We make the usual assumption (e.g., Weinberg, 1964; Nelson & Good, 1968; Mathews, 1971) that $a^{(\sigma)}(\mathbf{q})$ and $b^{(\sigma)}(\mathbf{q})$ are the annihilation operators for a particle and its antiparticle, respectively, and that $a^{(\sigma)*}(\mathbf{q})$ and $b^{(\sigma)*}(\mathbf{q})$ are the corresponding creation operators; the vector space on which these operators act has a positive definite metric (for the possibility of using states with an indefinite metric see, e.g., Pauli, 1950; Nagy, 1966). The operators $a^{(\sigma)}(\mathbf{q})$, and $b^{(\sigma)}(\mathbf{q})$ annihilate the vacuum state $|0\rangle$ and one has one of the two possibilities

$$[a^{(\sigma)}(\mathbf{q}), a^{(\sigma')*}(\mathbf{q}')]_{\pm} = [b^{(\sigma)}(\mathbf{q}), b^{(\sigma')*}(\mathbf{q}')]_{\pm} = \delta_{\sigma\sigma'}\delta(\mathbf{q} - \mathbf{q}') \tag{3.4a}$$

$$\begin{aligned}[a^{(\sigma)}(\mathbf{q}), a^{(\sigma')}(\mathbf{q}')]_{\pm} &= [b^{(\sigma)}(\mathbf{q}), b^{(\sigma')}(\mathbf{q}')]_{\pm} = [a^{(\sigma)*}(\mathbf{q}), a^{(\sigma')*}(\mathbf{q}')]_{\pm} \\ &= [b^{(\sigma)*}(\mathbf{q}), b^{(\sigma')*}(\mathbf{q}')]_{\pm} = 0\end{aligned} \tag{3.4b}$$

$$\begin{aligned}[a^{(\sigma)}(\mathbf{q}), b^{(\sigma')}(\mathbf{q}')]_{\pm} &= [a^{(\sigma)}(\mathbf{q}), b^{(\sigma')*}(\mathbf{q}')]_{\pm} = [a^{(\sigma)*}(\mathbf{q}), b^{(\sigma')*}(\mathbf{q}')]_{\pm} \\ &= [a^{(\sigma)*}(\mathbf{q}), b^{(\sigma')}(\mathbf{q}')]_{\pm} = 0\end{aligned} \tag{3.4c}$$

The upper sign yields Fermi statistics and the lower sign Bose statistics.

The scalar product in (2.16a) and (2.17a) becomes the charge operator

$$\begin{aligned}\bar{Q} &= \int d^3x \psi^{\dagger}(\mathbf{x}, t) M \psi(\mathbf{x}, t) - \int d^3x \langle 0 | \psi^{\dagger}(\mathbf{x}, t) M \psi(\mathbf{x}, t) | 0 \rangle \\ &= \int d^3x : \psi^{\dagger}(\mathbf{x}, t) M \psi(\mathbf{x}, t) :\end{aligned} \tag{3.5a}$$

and (2.17b) yields the field theory Hamiltonian

$$\begin{aligned}\bar{H} &= \int d^3x \psi^{\dagger}(\mathbf{x}, t) M H \psi(\mathbf{x}, t) - \int d^3x \langle 0 | \psi^{\dagger}(\mathbf{x}, t) M H \psi(\mathbf{x}, t) | 0 \rangle \\ &= \int d^3x : \psi^{\dagger}(\mathbf{x}, t) M H \psi(\mathbf{x}, t) :\end{aligned} \tag{3.5b}$$

where the infinite vacuum expectation terms have been subtracted. These expressions give the momentum space results

$$\bar{Q} = \begin{cases} \int d^3q \sum_{\sigma} [a^{(\sigma)*}(\mathbf{q}) a^{(\sigma)}(\mathbf{q}) + b^{(\sigma)}(\mathbf{q}) b^{(\sigma)*}(\mathbf{q}) - \langle 0 | b^{(\sigma)}(\mathbf{q}) b^{(\sigma)*}(\mathbf{q}) | 0 \rangle] & \text{if } M = I \quad (3.6a) \\ \int d^3q \sum_{\sigma} [a^{(\sigma)*}(\mathbf{q}) a^{(\sigma)}(\mathbf{q}) - b^{(\sigma)}(\mathbf{q}) b^{(\sigma)*}(\mathbf{q}) + \langle 0 | b^{(\sigma)}(\mathbf{q}) b^{(\sigma)*}(\mathbf{q}) | 0 \rangle] & \text{if } M = \rho_3 \quad (3.6b) \end{cases}$$

and

$$\bar{H} = \begin{cases} \int d^3qE(\mathbf{q}) \sum_\sigma [a^{(\sigma)*}(\mathbf{q})a^{(\sigma)}(\mathbf{q}) - b^{(\sigma)}(\mathbf{q})b^{(\sigma)*}(\mathbf{q}) + \langle 0 | b^{(\sigma)}(\mathbf{q})b^{(\sigma)*}(\mathbf{q}) | 0 \rangle] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } M = I \qquad (3.7a) \\ \int d^3qE(\mathbf{q}) \sum_\sigma [a^{(\sigma)*}(\mathbf{q})a^{(\sigma)}(\mathbf{q}) + b^{(\sigma)}(\mathbf{q})b^{(\sigma)*}(\mathbf{q}) \\ \qquad\qquad\qquad - \langle 0 | b^{(\sigma)}(\mathbf{q})b^{(\sigma)*}(\mathbf{q}) | 0 \rangle] \quad \text{if } M = \rho_3 \end{cases}$$

(3.7b)

In order for the energy of the states created by the operators $a^{(\sigma)*}(\mathbf{q})$ and $b^{(\sigma)*}(\mathbf{q})$ to be positive definite one must choose the anticommutator in (3.4) when $M = I$ and the commutator when $M = \rho_3$. Then, for either metric one has

$$\bar{Q} = \int d^3q \sum_\sigma [a^{(\sigma)*}(\mathbf{q})a^{(\sigma)}(\mathbf{q}) - b^{(\sigma)*}(\mathbf{q})b^{(\sigma)}(\mathbf{q})] \tag{3.8a}$$

$$\bar{H} = \int d^3qE(\mathbf{q}) \sum_\sigma [a^{(\sigma)*}(\mathbf{q})a^{(\sigma)}(\mathbf{q}) + b^{(\sigma)*}(\mathbf{q})b^{(\sigma)}(\mathbf{q})] \tag{3.8b}$$

For $G = \pm i$, in which case (1.1) is the Dirac-Hamiltonian, one has the familiar second quantization using Fermi statistics, but for real values of G we find that the theory is second quantized using Bose statistics.

The operator $\bar{H}$ is the generator of time translations in the second-quantized theory. One can similarly obtain the three-momentum $\bar{\mathbf{p}}$, the angular momentum $\bar{\mathbf{J}}$, and the boost operator $\bar{\mathbf{K}}$ from the expressions for the expectation values of the corresponding operators in the single-particle theory

$$\bar{\mathbf{p}} = \int d^3x\colon \psi^\dagger(\mathbf{x}, t)M\mathbf{p}\psi(\mathbf{x}, t)\colon \tag{3.9a}$$

$$\bar{\mathbf{J}} = \int d^3x\colon \psi^\dagger(\mathbf{x}, t)M\mathbf{J}\psi(\mathbf{x}, t)\colon \tag{3.9b}$$

$$\bar{\mathbf{K}} = \int d^3X\colon \psi^\dagger(\mathbf{x}, t)M\mathbf{K}\psi(\mathbf{x}, t)\colon \tag{3.9c}$$

where, as in (3.5), we use

$$\colon\psi^\dagger\mathcal{O}\psi\colon = \psi^\dagger\mathcal{O}\psi - \langle 0 | \psi^\dagger\mathcal{O}\psi | 0 \rangle \tag{3.10}$$

for any operator $\mathcal{O}$. Then, for infinitesimal proper orthochronous Poincaré transformations one should have, corresponding to (1.5)-(1.7),

$$(1 + i\mathbf{d} \cdot \bar{\mathbf{p}})\psi(\mathbf{x}, t)(1 - i\mathbf{d} \cdot \bar{\mathbf{p}}) = (1 - i\mathbf{d} \cdot \mathbf{p})\psi(\mathbf{x}, t) \tag{3.11a}$$

$$(1 - iD\bar{H})\psi(\mathbf{x}, t)(1 + iD\bar{H}) = (1 + IDH)\psi(\mathbf{x}, t) \tag{3.11b}$$

$$(1 + i\boldsymbol{\theta} \cdot \bar{\mathbf{J}})\psi(\mathbf{x}, t)(1 - i\boldsymbol{\theta} \cdot \bar{\mathbf{J}}) = (1 - i\boldsymbol{\theta} \cdot \mathbf{J})\psi(\mathbf{x}, t) \tag{3.11c}$$

$$(1 - i\boldsymbol{\lambda} \cdot \bar{\mathbf{K}})\psi(\mathbf{x}, t)(1 + i\boldsymbol{\lambda} \cdot \bar{\mathbf{K}}) = (1 + i\boldsymbol{\lambda} \cdot \mathbf{K})\psi(\mathbf{x}, t) \tag{3.11d}$$

The above relations may be verified from (3.8) and the corresponding momentum space expressions obtained from (3.9), or directly from the results of the next section for $[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)]_\pm$ and $[\psi(\mathbf{x}, t), \psi^*(x', t)]_\pm$.

According to (1.8) the charge conjugation operator $\overline{\mathscr{C}}$ should transform the field as follows:

$$\overline{\mathscr{C}}\psi(\mathbf{x}, t)\overline{\mathscr{C}}^{-1} = B\psi^*(\mathbf{x}, t) \tag{3.12}$$

where B is given by (1.9). Because of (2.18), (2.19), and (3.2) one has

$$Bu^{(\sigma)*}(\mathbf{q}) = \eta v^{(\sigma)}(\mathbf{q}) \tag{3.13a}$$

$$Bv^{(\sigma)*}(\mathbf{q}) = -\eta^* u^{(\sigma)}(\mathbf{q}) \tag{3.13b}$$

so, from (3.3) it follows that

$$\overline{\mathscr{C}}\psi(\mathbf{x}, t)\overline{\mathscr{C}}^{-1} = \frac{1}{(2\pi)^{3/2}}\int d^3q \sum_\sigma [\eta u^{(\sigma)}(\mathbf{q}) b^{(\sigma)}(\mathbf{q}) e^{i(\mathbf{q}\cdot\mathbf{x} - E(\mathbf{q})t)} - \eta^* v^{(\sigma)}(\mathbf{q}) a^{(\sigma)*}(\mathbf{q}) e^{-i(\mathbf{q}\cdot\mathbf{x} - E(\mathbf{q})t)}] \tag{3.14}$$

Consequently,

$$\overline{\mathscr{C}} a^{(\sigma)}(\mathbf{q})\overline{\mathscr{C}}^{-1} = \eta b^{(\sigma)}(\mathbf{q}) \tag{3.15a}$$

$$\overline{\mathscr{C}} b^{(\sigma)}(\mathbf{q})\overline{\mathscr{C}}^{-1} = -\eta a^{(\sigma)}(\mathbf{q}) \tag{3.15b}$$

Thus, we have the interesting property

$$\overline{\mathscr{C}}^2 a^{(\sigma)}(\mathbf{q})\overline{\mathscr{C}}^{-2} = a^{(\sigma)}(\mathbf{q}) \tag{3.16a}$$

if $G = \pm i$, but

$$\overline{\mathscr{C}}^2 a^{(\sigma)}(\mathbf{q})\overline{\mathscr{C}}^{-2} = -a^{(\sigma)}(\mathbf{q}) \tag{3.16b}$$

if G is real.

4. *Locality*

For the fields introduced in the preceding section one has, as a result of (3.3) and (3.4),

$$[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t')]_+ = 0 \tag{4.1a}$$

if $G = \pm i$ and

$$[\psi(\mathbf{x}, t), \psi(\mathbf{x}', t')]_- = 0 \tag{4.1b}$$

if G is real. In manifestly covariant theories a quantized field is said to be *local* (e.g., Streater and Wightman, 1964; Wightman, 1973) if, in addition to (4.1a),

$$[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')]_+ = 0 \tag{4.2a}$$

or if, in addition to (4.1b),

$$[\psi(\mathbf{x}, t), \psi^*(\mathbf{x}', t')]_- = 0 \tag{4.2b}$$

whenever the interval between the space-time points $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$ is spacelike. For the Dirac field (our case $G = \pm i$), (4.2a) is valid for intervals outside the light cone; in the following we shall demonstrate that, for our nonmanifestly covariant equation with real values of G (4.2b) is valid.

According to (3.3) and (3.4), one has, for $r, s = 1, 2, 3, 4$,

$$[\psi_r(\mathbf{x}, t), \psi_s{}^*(\mathbf{x}', t')]_\pm$$

$$= \frac{1}{(2\pi)^3} \int d^3q \sum_\sigma [u_r^{(\sigma)*}(\mathbf{q}) u_s^{(\sigma)*}(\mathbf{q}) e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x})'} e^{-iE(\mathbf{q})(t-t')}$$

$$\pm v_r^{(\sigma)}(\mathbf{q}) v_s^{(\sigma)*}(\mathbf{q}) e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} e^{iE(\mathbf{q})(t-t')}] \tag{4.3}$$

With the aid of (2.1a), (2.2), (2.14), and (3.2) we find that

$$\sum_\sigma u_r^{(\sigma)}(\mathbf{q}) u_s^{(\sigma)*}(\mathbf{q}) = \frac{1}{2E(\mathbf{q})} \{[E(\mathbf{q}) + H(\mathbf{q})]M\}_{rs} \tag{4.4a}$$

$$\sum_\sigma v_r^{(\sigma)}(\mathbf{q}) v_s^{(\sigma)*}(\mathbf{q}) = \frac{\epsilon_M}{2E(\mathbf{q})} \{[E(\mathbf{q}) - H(-\mathbf{q})]M\}_{rs} \tag{4.4b}$$

where

$$\epsilon_M = \begin{cases} 1 & \text{if } M = I \\ -1 & \text{if } M = \rho_3 \end{cases} \tag{4.5}$$

When (4.4) is substituted into (4.3) the result is

$$[\psi_r(\mathbf{x}, t), \psi_s^*(x', t')]_\pm$$

$$= \left\{\left[i\frac{\partial}{\partial t} + H(-i\nabla)\right]M\right\}_{rs} \frac{1}{2(2\pi)^3} \int \frac{d^3q}{E(\mathbf{q})} \{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} e^{-iE(\mathbf{q})(t-t')}$$

$$\mp \epsilon_M e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} e^{iE(\mathbf{q})(t-t')}\} \tag{4.6}$$

For the cases we are considering (i.e., the anticommutator when $G = \pm i$ and the commutator when G is real), one has $\mp\epsilon_M = -1$, so, as is well known (e.g., Pauli, 1940) the integral on the right-hand side of (4.6) vanishes when the interval between $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$ lies outside the light cone; since the operator $i\partial/\partial t + H(-i\nabla)$ is local the right-hand side of (4.6) vanishes for such points. Thus, for real values of G the commutator of the field and its adjoint vanishes if the interval between their arguments is spacelike, and one can consider the theory to be local in this sense.

Before proceeding, let us comment that for equal times (4.6) becomes

$$[\psi_r(\mathbf{x}, t), \psi_s^*(\mathbf{x}', t')]_\pm = M_{rs}\delta(\mathbf{x} - \mathbf{x}') \tag{4.7}$$

It is now a simple matter to verify equations (3.11), using the definitions (3.5b) and (3.9) for the generators of proper orthochronous Poincaré transformations and also using (4.1) and (4.7).

We must be cautious about concluding that one can actually base a local theory on our equation with real G, because the locality requirement (4.1) for the fields is generally made for manifestly covariant theories.[3] In such theories one can construct observables from the fields and a finite number of their derivatives, so (4.1) and (4.2) are sufficient conditions for two such observables to commute when the interval between their arguments lies outside the light cone. It is not clear whether one can construct observables with the desired property for the theory developed here for real values of G, particularly since the boost generator **K** in the single-particle theory apparently is not local. As an illustration of the problems involved, one can consider the possibility of finding a local four-current whose fourth component, when integrated over all space, is equal to the charge operator (3.6a).

It has been shown (Guertin, 1975b) that for real G the continuity equation

$$\partial\rho/\partial t = -\nabla\cdot\mathbf{j} \tag{4.8}$$

is satisfied by

$$\rho = \tfrac{1}{2}\colon[(V\psi)^\dagger\rho_3(U\psi) + (U\psi)^\dagger\rho_3(V\psi)]\colon \tag{4.9a}$$

$$\begin{aligned}\mathbf{j} = &-[i(1+G^2)/4m]\colon[(V\psi)^\dagger(1+\rho_1)(\nabla U\psi) + (U\psi)^\dagger(1+\rho_1)(\nabla V\psi)\\ &-(\nabla U\psi)^\dagger(1+\rho_1)(V\psi) - (\nabla V\psi)^\dagger(1+\rho_1)(U\psi)]\colon\\ &-\tfrac{1}{2}G\colon[(V\psi)^\dagger\rho_2\sigma(V\psi) + (U\psi)^\dagger\rho_2\boldsymbol{\sigma}(V\psi)]\colon\end{aligned} \tag{4.9b}$$

for any choice of operators U and V with the properties

$$[U, H]_- = [V, H]_- = 0 \tag{4.10a}$$

$$V^\dagger\rho_3 U + U^\dagger\rho_3 V = 2\rho_3 \tag{4.10b}$$

Then,

$$\int d^3x\rho(\mathbf{x}, t) = \bar{Q} \tag{4.11}$$

as desired, and, if U and V are local operators, the relations

$$[\rho(\mathbf{x}, t), \rho(\mathbf{x}', t')]_- = 0 \tag{4.12a}$$

$$[\mathbf{j}(\mathbf{x}, t), \mathbf{j}(\mathbf{x}', t')]_- = 0 \tag{4.12b}$$

$$[\rho(\mathbf{x}, t), \mathbf{j}(\mathbf{x}', t')]_- = 0 \tag{4.12c}$$

are certainly satisfied whenever the interval between $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$ is space-like. But, in general, and in particular for $U = V = 1$, ρ and $\mathbf{j}$ are not the components of a four-vector and cannot be a satisfactory charge and current density. A question which the author is unable to answer at this time is whether one can find a U and a V such that ρ and $\mathbf{j}$ have all the desired properties.

[3] In fact, the reason that the vanishing of the commutator (4.2b) outside the light cone for real values of G does not contradict the usual proofs of the connection between spin and statistics (e.g., Pauli, 1940; Streater and Wightman, 1964) is that manifest covariance of the fields has been one of the assumptions upon which such proofs have been based.

5. *Summary*

We have second quantized the Schrödinger equation (1.4) using the Hamiltonian (1.1) for massive spin-$\frac{1}{2}$ particles, subject to the conditions that there be both a positive definite metric and a positive definite energy in the Fock space in which the field and its adjoint operate. In contrast to the case $G = \pm i$ (the Dirac equation), it was concluded that for real values of G one should employ Bose statistics, and it was then found that the commutator of the field and its adjoint vanishes when the space-time interval between their arguments lies outside the light cone.

The question as to whether one actually can construct a local field theory remains unanswered, because we have been unable to demonstrate that there exist observables that commute outside the light cone. It may happen that a satisfactory theory is possible for particular real values of G and for particular values of the variable $\theta(p)$ in (2.8) and (2.9). Even if the above problem can be treated in an acceptable manner, one must still consider the possibility of inconsistencies when interactions are introduced, a problem that exists even in the case of manifestly covariant equations (e.g., Wightman, 1971, 1972, 1973).

References

Foldy, L. L. (1956). *Physical Review*, **102**, 568.

Fuschich, V. I., Grischenko, A. L., and Nikitin, A. G. (1971). *Teoreticheskaya i Matematicheskaya Fizika*, 8, 192 (English translation: *Theoretical and Mathematical Physics*, 8, 766.)

Guertin, R. F. (1974). *Annals of Physics (New York)*, 88, 504.

Guertin, R. F. (1975a). *Annals of Physics (New York)*, **91**, 386.

Guertin, R. F. (1975b). *International Journal of Theoretical Physics*, **14**, 385.

Heitler, W. (1943). *Proceedings of the Royal Irish Academy*, **49**, 1.

Kolsrud, M. (1971). *Physica Norvegica*, **5**, 169.

Mathews, P. M. (1971). "Wave Equations of Arbitrary Spin: Relativistic Invariance and Quantization," in *Lectures in Theoretical Physics*, K. T. Mahanthappa and W. E. Brittin, eds., Vol. XII C, p. 139. Gordon and Breach, New York.

Nagy, K. L. (1966). *State Vector Spaces with Indefinite Metric in Quantum Field Theory*. P. Noordhoff, Groningen.

Nelson, T. J. and Good, R. H. (1968). *Reviews of Modern Physics*, **40**, 508.

Pauli, W. (1940). *Physical Review*, **58**, 716.

Pauli, W. (1950). *Progress of Theoretical Physics (Japan)*, **5**, 526.

Sakata, S. and Taketani, M. (1940). *Proceedings of the Physico-Mathematical Society of Japan*, **22**, 757.

Streater, R. F. and Wightman, A. S. (1964). *PCT, Spin and Statistics and All That*, Chap. 4. W. A. Benjamin, New York.

Weinberg, S. (1964). *Physical Review*, **133**, B1318.

Wightman, A. S. (1971). "Introductory Remarks," in *Troubles in the External Field Problem for Invariant Wave Equations*, p. 1. Gordon and Breach, New York.

Wightman, A. S. (1972). "The Dirac Equation," in *Aspects of Quantum Theory*, A. Salam and E. P. Wigner, eds., p. 95. Cambridge University Press.

Wightman, A. S. (1973). "Relativistic Wave Equations as Singular Hyperbolic Systems," in *Proceedings of Symposia in Pure Mathematics, Vol. 23, Partial Differential Equations*, D. C. Spencer, ed., p. 441. American Mathematical Society, Providence, Rhode Island.